

A PLACTIC ALGEBRA OF EXTREMAL WEIGHT CRYSTALS AND THE CAUCHY IDENTITY FOR SCHUR OPERATORS

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ABSTRACT. We give a new bijective interpretation of the Cauchy identity for Schur operators which is a commutation relation between two formal power series with operator coefficients. We introduce a plactic algebra associated with the Kashiwara's extremal weight crystals over the Kac-Moody algebra of type $A_{+\infty}$, and construct a Knuth type correspondence preserving the plactic relations. This bijection yields the Cauchy identity for Schur operators as a homomorphic image of its associated identity for plactic characters of extremal weight crystals, and also recovers the Sagan and Stanley's correspondence for skew tableaux as its restriction.

1. INTRODUCTION

Let $\Lambda = \Lambda_{\mathbf{x}}$ be the algebra of symmetric functions in formal commuting variables $\mathbf{x} = \{x_1, x_2, \dots\}$ over \mathbb{Q} . We denote by \mathcal{P} the set of partitions and let $s_{\lambda}(\mathbf{x})$ be the Schur function in \mathbf{x} corresponding to $\lambda \in \mathcal{P}$. Let

$$\mathcal{P}(\mathbf{x}) = \sum_{\lambda \in \mathcal{P}} s_{\lambda} s_{\lambda}(\mathbf{x}), \quad \mathcal{Q}(\mathbf{x}) = \sum_{\lambda \in \mathcal{P}} s_{\lambda}^{\perp} s_{\lambda}(\mathbf{x}) \in \text{End}_{\mathbb{Q}}(\Lambda)[[\mathbf{x}]],$$

where s_{λ} and s_{λ}^{\perp} are linear operators on Λ induced from the left multiplication by $s_{\lambda}(\mathbf{x})$ and its adjoint with respect to the Hall inner product on Λ , respectively. One may regard s_{λ} and s_{λ}^{\perp} as operators on $\mathbb{Q}\mathcal{P} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q}\lambda$, where λ is identified with $s_{\lambda}(\mathbf{x})$. Moreover s_{λ} and s_{λ}^{\perp} can be given as Schur functions in certain locally non-commutative operators on $\mathbb{Q}\mathcal{P}$ called *Schur operators* by Fomin, while $\mathcal{P}(\mathbf{x})$ and $\mathcal{Q}(\mathbf{y})$ can be written as Cauchy products in Schur operators and \mathbf{x} [3, 4].

Let $\mathbf{y} = \{y_1, y_2, \dots\}$ be another formal commuting variables. It is well known that the following commutation relation holds;

$$(1.1) \quad \mathcal{Q}(\mathbf{y})\mathcal{P}(\mathbf{x}) = \mathcal{P}(\mathbf{x})\mathcal{Q}(\mathbf{y}) \frac{1}{\prod_{i,j}(1 - x_i y_j)}$$

called *generalized Cauchy identity* or *Cauchy identity for Schur operators*. Considering both sides as operators with coefficients in $\Lambda_{\mathbf{x}} \otimes \Lambda_{\mathbf{y}}$ and then equating each entry of their matrix forms, we obtain a Cauchy identity for skew Schur functions

[15]

$$\sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda/\beta}(\mathbf{y}) = \sum_{\eta} s_{\beta/\eta}(\mathbf{x}) s_{\alpha/\eta}(\mathbf{y}) \frac{1}{\prod_{i,j} (1 - x_i y_j)},$$

where α, β are given partitions. A bijective interpretation of the Cauchy identity for skew Schur functions was given by Sagan and Stanley [16], and it was extended to a bijection in a more general framework by Fomin [3] including various analogues of Knuth correspondence.

Recently, a new representation theoretic interpretation of the Cauchy identity for Schur operators was given by the author [10] using the notion of Kashiwara's extremal weight crystals [8] over the quantized enveloping algebra associated with the Kac-Moody algebra of type $A_{+\infty}$, say $\mathfrak{gl}_{>0}$. It is proved that a Schur operator can be realized as a functor of tensoring by an extremal weight crystal element and (1.1) can be understood (but not in a bijective way) as a non-commutative character identity corresponding to the decomposition of the crystal graph of the Fock space with infinite positive level, which is an infinite analogue of the level n fermionic Fock space decomposition due to Frenkel [5].

Motivated by a categorification of Schur operators in [10], we give a new combinatorial way to explain both the Cauchy identity for Schur operators and skew Schur functions in terms of a single bijection. More precisely, the main result in this paper is to construct a Knuth type correspondence, which gives a bijective interpretation of the identity (1.1) or its dual form, as the usual Knuth correspondence does for the Cauchy product, and also recovers Sagan and Stanley's correspondence as a restriction of this bijection.

Our approach is to define a t -analogue of the plactic algebra $\mathcal{U}(t)$ for $\mathfrak{gl}_{>0}$ generated by u_i and u_{i^\vee} for $i \geq 1$ with t an indeterminate, where the subalgebra generated by u_i (resp. u_{i^\vee}) is isomorphic to the usual plactic algebra by Lascoux and Schützenberger [13]. We show that $\mathcal{U}(1)$ is isomorphic to the plactic algebra defined by using the notion of crystal equivalence (cf.[14]). Note that each monomial in $\mathcal{U}(1)$ corresponds in general to an element of an extremal weight crystal, which may not be either highest or lowest weight crystal.

Now, let $\mathcal{M}_{\mathcal{A}, \mathcal{B}}$ be the set of $\mathcal{A} \times \mathcal{B}$ matrices $A = (a_{ij})$ with entries in $\mathbb{Z}_{\geq 0}$ such that $\sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{B}} a_{ij} < \infty$ and $a_{ij} \leq 1$ for $|i| \neq |j|$, where \mathcal{A} and \mathcal{B} are arbitrary \mathbb{Z}_2 -graded sets and $|\cdot|$ denotes the degree of an element in \mathcal{A} or \mathcal{B} . We assume that all the elements in \mathbb{N} and \mathbb{N}^\vee are of degree 0. By using the usual Knuth map and non-commutative Littlewood-Richardson rule for extremal weight crystals for $\mathfrak{gl}_{>0}$ [10, 11], we construct an explicit bijection (Theorem 5.1);

$$\mathcal{M}_{\mathcal{A}, \mathbb{N}} \times \mathcal{M}_{\mathcal{B}, \mathbb{N}^\vee} \longrightarrow \mathcal{M}_{\mathcal{B}, \mathbb{N}^\vee} \times \mathcal{M}_{\mathcal{A}, \mathbb{N}} \times \mathcal{M}_{\mathcal{A}, \mathcal{B}},$$

which preserves the weights with respect to \mathcal{A} and \mathcal{B} , and the plactic relations of $\mathcal{U}(t)$ for the column words with entries in $\mathbb{N} \cup \mathbb{N}^\vee$ on both sides. As a corollary, we obtain a character identity in locally non-commuting variables $\mathbf{u} = \{u_i, u_{i^\vee} \mid i \geq 1\}$ and commuting variables $\mathbf{x}_\mathcal{A} = \{x_a \mid a \in \mathcal{A}\}$, $\mathbf{x}_\mathcal{B} = \{x_b \mid b \in \mathcal{B}\}$ (Corollary 5.2). In particular, when $\mathcal{A} = \mathcal{B} = \mathbb{N}$, this identity recovers (1.1) under a homomorphism sending u_i and u_{i^\vee} to Schur operators on $\mathbb{Q}\mathcal{P}$ and specializing $t = 1$. Moreover, the Knuth correspondence for skew tableaux by Sagan and Stanley can be recovered by restricting the above bijection to the pairs of matrices on the left-hand side whose column words are Littlewood-Richardson words of shape (α, β) with $\alpha, \beta \in \mathcal{P}$ (see Section 5.3 for a definition).

The paper is organized as follow. In Section 2, we briefly recall necessary background for semistandard tableaux and Knuth correspondence. In Section 3, we recall the notion of rational semistandard tableaux for $\mathfrak{gl}_{>0}$ and their insertion algorithm. In Section 4, we introduce a plactic algebra for $\mathfrak{gl}_{>0}$ associated with rational semistandard tableaux. Finally, in Section 5, we construct a Knuth type correspondence and its associated non-commutative character identity.

2. PRELIMINARY

2.1. Semistandard tableaux. Throughout this paper, we assume that \mathcal{A} (or \mathcal{B}) is a linearly ordered \mathbb{Z}_2 -graded set, that is, $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$, which is at most countable. We usually denote by $<$ a linear ordering on a given linearly ordered \mathbb{Z}_2 -graded set. For $a \in \mathcal{A}_\epsilon$ ($\epsilon \in \mathbb{Z}_2$), we put $|a| = \epsilon$. By convention, we let $\mathbb{N} = \{1 < 2 < \cdots\}$, and $[n] = \{1 < \cdots < n\}$ for $n \geq 1$, where all the elements are of degree 0.

Let \mathcal{P} denote the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i \geq 1}$ with a Young diagram or a subset $\{(i, j) \mid 1 \leq j \leq \lambda_i\}$ of $\mathbb{N} \times \mathbb{N}$ following [15]. Let $\ell(\lambda) = |\{i \mid \lambda_i \neq 0\}|$. We denote by $\lambda' = (\lambda'_i)_{i \geq 1}$ the conjugate partition of λ whose Young diagram is $\{(i, j) \mid (j, i) \in \lambda\}$. For $\mu \in \mathcal{P}$ with $\lambda \supset \mu$, λ/μ denotes the skew Young diagram.

For a skew Young diagram λ/μ , a tableau T obtained by filling λ/μ with entries in \mathcal{A} is called \mathcal{A} -semistandard if (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in \mathcal{A}_0 (resp. \mathcal{A}_1) are strictly increasing in each column (resp. row). We say that λ/μ is the shape of T , and write $\text{sh}(T) = \lambda/\mu$.

We denote by $SST_\mathcal{A}(\lambda/\mu)$ the set of all \mathcal{A} -semistandard tableaux of shape λ/μ . We set $\mathcal{P}_\mathcal{A} = \{\lambda \in \mathcal{P} \mid SST_\mathcal{A}(\lambda) \neq \emptyset\}$. For example, $\mathcal{P}_\mathcal{A} = \mathcal{P}$ when \mathcal{A} is an infinite set, and $\mathcal{P}_{[n]} = \{\lambda \mid \ell(\lambda) \leq n\}$.

Let $\mathcal{W}_\mathcal{A}$ be the set of finite words with letters in \mathcal{A} . For $T \in SST_\mathcal{A}(\lambda/\mu)$, we denote by $T(i, j)$ the entry of T at $(i, j) \in \lambda/\mu$. We denote by $w_{\text{col}}(T) \in \mathcal{W}_\mathcal{A}$ the

word obtained by reading the entries of T column by column from right to left, and from top to bottom in each column. Also, we denote by $w_{\text{row}}(T) \in \mathcal{W}_{\mathcal{A}}$ the word obtained by reading the entries of T row by row from top to bottom, and from right to left in each row.

Let $P_{\mathcal{A}} = \bigoplus_{a \in \mathcal{A}} \mathbb{Z}\epsilon_a$ be the free abelian group with the basis $\{\epsilon_a \mid a \in \mathcal{A}\}$ and let $\mathbf{x}_{\mathcal{A}} = \{x_a \mid a \in \mathcal{A}\}$ be a set of commuting formal variables. For $\lambda = \sum_{a \in \mathcal{A}} \lambda_a \epsilon_a \in P_{\mathcal{A}}$, let $\mathbf{x}_{\mathcal{A}}^{\lambda} = \prod_{a \in \mathcal{A}} x_a^{\lambda_a}$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathcal{A}}$, we define $\text{wt}_{\mathcal{A}}(w) = \sum_{1 \leq i \leq r} \epsilon_{w_i} \in P_{\mathcal{A}}$. For a skew Young diagram λ/μ and $T \in SST_{\mathcal{A}}(\lambda/\mu)$, we define $\text{wt}_{\mathcal{A}}(T) = \sum_{(i,j) \in \lambda/\mu} \epsilon_{T(i,j)}$. Let $s_{\lambda/\mu}(\mathbf{x}_{\mathcal{A}}) = \sum_{T \in SST_{\mathcal{A}}(\lambda/\mu)} \mathbf{x}_{\mathcal{A}}^{\text{wt}_{\mathcal{A}}(T)}$, which is the character of $SST_{\mathcal{A}}(\lambda/\mu)$. Note that $s_{\lambda/\mu}(\mathbf{x}_{\mathbb{N}})$ is a usual (skew) Schur function when $\mathcal{A} = \mathbb{N}$.

We will also use the following operations on tableaux;

- (1) *dual* : Let $\mathcal{A}^{\vee} = \{a^{\vee} \mid a \in \mathcal{A}\}$ be the linearly ordered \mathbb{Z}_2 -graded set with $|a^{\vee}| = |a|$ and $a^{\vee} < b^{\vee}$ for $a > b$. For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, we define T^{\vee} to be the tableau obtained by applying 180°-rotation to T and replacing each entry a in T with a^{\vee} . Then $T^{\vee} \in SST_{\mathcal{A}^{\vee}}((\lambda/\mu)^{\vee})$, where $(\lambda/\mu)^{\vee}$ denotes the shape of T^{\vee} . We use the convention that $(a^{\vee})^{\vee} = a$ for $a \in \mathcal{A}$ and hence $(T^{\vee})^{\vee} = T$.
- (2) *gluing* : Let $\mathcal{A} * \mathcal{B}$ be the \mathbb{Z}_2 -graded set $\mathcal{A} \sqcup \mathcal{B}$ with the extended linearly ordering given by $a < b$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{B}}(\lambda/\mu)$, we define $S * T \in SST_{\mathcal{A} * \mathcal{B}}(\lambda)$ by $S * T(i, j) = S(i, j)$ for $(i, j) \in \mu$ and $T(i, j)$ for $(i, j) \in \lambda/\mu$.

2.2. Littlewood-Richardson rule. For $a \in \mathcal{A}$ and $T \in SST_{\mathcal{A}}(\lambda)$ with $\lambda \in \mathcal{P}_{\mathcal{A}}$, $a \rightarrow T$ (resp. $T \leftarrow a$) denotes the tableau obtained by the Schensted column (resp. row) insertion (see for example, [6, Appendix A.2] and [1] for its super-analogue). For $w = w_1 \dots w_r \in \mathcal{W}_{\mathcal{A}}$, we let $(w \rightarrow T) = (w_r \rightarrow (\dots (w_1 \rightarrow T) \dots))$ and $(T \leftarrow w) = ((\dots (T \leftarrow w_1) \dots) \leftarrow w_r)$. For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{A}}(\nu)$ with $\mu, \nu \in \mathcal{P}_{\mathcal{A}}$, we define $T \rightarrow S$ (resp. $S \leftarrow T$) by $(w(T)_{\text{col}} \rightarrow S)$ (resp. $S \leftarrow (w(T)_{\text{row}})^{\text{rev}}$), where w^{rev} denotes the reverse word of w .

For $\lambda, \mu, \nu \in \mathcal{P}$ with $|\lambda| = |\mu| + |\nu|$, let $\mathbf{LR}_{\mu\nu}^{\lambda}$ be the set of tableaux U in $SST_{\mathbb{N}}(\lambda/\mu)$ such that

- (1) $\text{wt}_{\mathbb{N}}(U) = \sum_{i \geq 1} \nu_i \epsilon_i$,
- (2) for $1 \leq k \leq |\nu|$, the number of occurrences of each $i \geq 1$ in $w_1 \dots w_k$ is no less than that of $i + 1$ in $w_1 \dots w_k$, where $w(U)_{\text{col}} = w_1 \dots w_{|\nu|}$.

We call $\mathbf{LR}_{\mu\nu}^{\lambda}$ the set of Littlewood-Richardson tableaux of shape λ/μ with content ν and put $c_{\mu\nu}^{\lambda} = |\mathbf{LR}_{\mu\nu}^{\lambda}|$ [15]. Let us introduce a variation of $\mathbf{LR}_{\mu\nu}^{\lambda}$, which is necessary for our later arguments. Define $\overline{\mathbf{LR}}_{\mu\nu}^{\lambda}$ to be the set of tableaux U in $SST_{-\mathbb{N}}(\lambda/\mu)$ such that

- (1) $\text{wt}_{-\mathbb{N}}(U) = \sum_{i \geq 1} \nu_i \epsilon_{-i}$,
- (2) for $1 \leq k \leq |\nu|$, the number of occurrences of each $-i \leq -1$ in $w_k \dots w_{|\nu|}$ is no less than that of $-(i+1)$ in $w_k \dots w_{|\nu|}$, where $w(U)_{\text{col}} = w_1 \dots w_{|\nu|}$.

Note that for $U \in SST_{\mathbb{N}}(\lambda/\mu)$, $U \in \mathbf{LR}_{\mu\nu}^{\lambda}$ if and only if U is Knuth equivalent to $H_{\nu} \in SST_{\mathbb{N}}(\nu)$, where $H_{\nu}(i, j) = i$ for $(i, j) \in \nu$ (cf.[6]). Similarly, we have for $U \in SST_{-\mathbb{N}}(\lambda/\mu)$, $U \in \overline{\mathbf{LR}}_{\mu\nu}^{\lambda}$ if and only if U is Knuth equivalent to $L_{\nu} \in SST_{-\mathbb{N}}(\nu)$, where $L_{\nu}(i, j) = -\nu'_j + i - 1$ for $(i, j) \in \nu$.

There is also a one-to-one correspondence from the set of $V \in SST_{\mathbb{N}}(\nu)$ such that $(V \rightarrow H_{\mu}) = H_{\lambda}$ to $\mathbf{LR}_{\mu\nu}^{\lambda}$. Indeed, V corresponds to $\iota(V) \in \mathbf{LR}_{\mu\nu}^{\lambda}$ where the number of k 's in the i -th row of V is equal to the number of i 's in the k -th row of $\iota(V)$ for $i, k \geq 1$.

Example 2.1.

$$SST_{\mathbb{N}}((3, 3, 2)) \ni \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & \end{array} \xrightarrow{\iota} \begin{array}{cccc} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 1 & 2 & 2 & \\ 2 & 3 & & & \\ 3 & & & & \end{array} \in \mathbf{LR}_{(3,1) \ (3,3,2)}^{(5,4,2,1)}$$

For $S \in SST_{\mathcal{A}}(\mu)$ and $T \in SST_{\mathcal{A}}(\nu)$, suppose that $\lambda = \text{sh}(T \rightarrow S)$ and $w_{\text{col}}(T) = w_1 \dots w_r$. Define $(T \rightarrow S)_R$ to be the tableau of shape λ/μ such that $\text{sh}(w_1 \dots w_k \rightarrow S) / \text{sh}(w_1 \dots w_{k-1} \rightarrow S)$ is filled with i if w_k is in the i th row of T for $1 \leq k \leq r$. Then the map $(S, T) \mapsto ((T \rightarrow S), (T \rightarrow S)_R)$ gives a bijection [19]

$$(2.1) \quad SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{A}}(\nu) \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}}} SST_{\mathcal{A}}(\lambda) \times \mathbf{LR}_{\mu\nu}^{\lambda},$$

which also implies $s_{\mu}(\mathbf{x}_{\mathcal{A}})s_{\nu}(\mathbf{x}_{\mathcal{A}}) = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}(\mathbf{x}_{\mathcal{A}})$.

2.3. Skew Littlewood-Richardson rule. Let λ/μ be a skew Young diagram. Let U be a tableau of shape λ/μ with entries in $\mathcal{A} \sqcup \mathcal{B}$, satisfying the following conditions;

- (S1) $U(i, j) \leq U(i', j')$ whenever $U(i, j), U(i', j') \in \mathcal{X}$ for $(i, j), (i', j') \in \lambda/\mu$ with $i \leq i'$ and $j \leq j'$,
- (S2) in each column of U , entries in \mathcal{X}_0 increase strictly from top to bottom,
- (S3) in each row of U , entries in \mathcal{X}_1 increase strictly from left to right,

where $\mathcal{X} = \mathcal{A}$ or \mathcal{B} . Suppose that $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are two adjacent entries in U such that b is placed above or to the left of a . Interchanging a and b is called a switching if the resulting tableau still satisfies the conditions (S1), (S2) and (S3).

For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, let U be a tableau obtained from $H_{\mu} * T$ by applying switching procedures as far as possible (in this case, $\mathcal{B} = \mathbb{N}$). Then $U = j(T) * j(T)_R$ for some $j(T) \in SST_{\mathcal{A}}(\nu)$ and $j(T)_R \in SST_{\mathbb{N}}(\lambda/\nu)$ with $\nu \in \mathcal{P}$. Then by [2,

Theorem 3.1], the map sending T to $(\mathcal{I}(T), \mathcal{I}(T)_R)$ gives a bijection

$$(2.2) \quad SST_{\mathcal{A}}(\lambda/\mu) \longrightarrow \bigsqcup_{\nu \in \mathcal{P}_{\mathcal{A}}} SST_{\mathcal{A}}(\nu) \times \mathbf{LR}_{\nu\mu}^{\lambda}.$$

In particular, the map $Q \mapsto \mathcal{I}(Q)_R$ restricts to a bijection from $\mathbf{LR}_{\mu\nu}^{\lambda}$ to $\mathbf{LR}_{\nu\mu}^{\lambda}$, and from $\overline{\mathbf{LR}}_{\mu\nu}^{\lambda}$ to $\mathbf{LR}_{\nu\mu}^{\lambda}$ when $\mathcal{A} = \pm\mathbb{N}$, respectively.

2.4. RSK correspondence. Let $\mathcal{M}_{\mathcal{A},\mathcal{B}}$ be the set of $\mathcal{A} \times \mathcal{B}$ matrices $A = (a_{ij})$ with entries in $\mathbb{Z}_{\geq 0}$ such that $\sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{B}} a_{ij} < \infty$ and $a_{ij} \leq 1$ for $|i| \neq |j|$. Let $\Omega_{\mathcal{A},\mathcal{B}}$ be the set of biwords $(\mathbf{i}, \mathbf{j}) \in \mathcal{W}_{\mathcal{A}} \times \mathcal{W}_{\mathcal{B}}$ such that

- (1) $\mathbf{i} = i_1 \cdots i_r$ and $\mathbf{j} = j_1 \cdots j_r$ for some $r \geq 0$,
- (2) $(i_1, j_1) \leq \cdots \leq (i_r, j_r)$,
- (3) $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $|i_s| \neq |j_s|$ for $1 \leq s < r$,

where for (i, j) and $(k, l) \in \mathcal{A} \times \mathcal{B}$,

$$(i, j) < (k, l) \iff \begin{cases} (i < k) & \text{or,} \\ (i = k, |i| = 0, \text{ and } j > l) & \text{or,} \\ (i = k, |i| = 1, \text{ and } j < l) & . \end{cases}$$

There is a bijection from $\Omega_{\mathcal{A},\mathcal{B}}$ to $\mathcal{M}_{\mathcal{A},\mathcal{B}}$, where (\mathbf{i}, \mathbf{j}) is mapped to $A(\mathbf{i}, \mathbf{j}) = (a_{ij})$ with $a_{ij} = |\{k \mid (i_k, j_k) = (i, j)\}|$. Note that the pair of empty words (\emptyset, \emptyset) corresponds to zero matrix.

For $A = A(\mathbf{i}, \mathbf{j}) \in \mathcal{M}_{\mathcal{A},\mathcal{B}}$, we let $P(A) = (\mathbf{j} \rightarrow \emptyset)$ and let $Q(A)$ be the tableau of the same shape as $P(A)$ such that $\text{sh}(j_1 \dots j_k \rightarrow \emptyset) / \text{sh}(j_1 \dots j_{k-1} \rightarrow \emptyset)$ is filled with i_k for $k \geq 1$. Then the map sending A to $(P(A), Q(A))$ gives a bijection

$$(2.3) \quad \mathcal{M}_{\mathcal{A},\mathcal{B}} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}}} SST_{\mathcal{B}}(\lambda) \times SST_{\mathcal{A}}(\lambda),$$

which is the (super analogue of) RSK correspondence [9]. If we define $\text{wt}_{\mathcal{A}}(A) = \text{wt}_{\mathcal{A}}(\mathbf{j})$ and $\text{wt}_{\mathcal{B}}(A) = \text{wt}_{\mathcal{A}}(\mathbf{i})$, then the bijection preserves $\text{wt}_{\mathcal{A}}$ and $\text{wt}_{\mathcal{B}}$. In terms of characters, we obtain the following Cauchy identity

$$\frac{\prod_{|a| \neq |b|} (1 + x_a x_b)}{\prod_{|a| = |b|} (1 - x_a x_b)} = \sum_{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}}} s_{\lambda}(\mathbf{x}_{\mathcal{B}}) s_{\lambda}(\mathbf{x}_{\mathcal{A}}),$$

where $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Similarly, for $A = (a_{ij}) \in \mathcal{M}_{\mathcal{A},\mathcal{B}}$, let $A' = (a'_{ij^{\vee}})$ be the unique matrix in $\mathcal{M}_{\mathcal{A},\mathcal{B}^{\vee}}$ such that $a_{ij} = a'_{i,j^{\vee}}$ for $(i, j) \in \mathcal{A} \times \mathcal{B}$. Then the map sending A to $(P(A')^{\vee}, Q(A'))$ gives a bijection

$$(2.4) \quad \mathcal{M}_{\mathcal{A},\mathcal{B}} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}}} SST_{\mathcal{B}}(\lambda^{\vee}) \times SST_{\mathcal{A}}(\lambda).$$

Finally, for $\mu \in \mathcal{P}_{\mathcal{B}}$, we have

$$\begin{aligned}
& SST_{\mathcal{B}}(\mu) \times \mathcal{M}_{\mathcal{A}, \mathcal{B}} \\
& \xleftarrow{1-1} \bigsqcup_{\mu \in \mathcal{P}_{\mathcal{B}} \nu \in \mathcal{P}_{\mathcal{A}}} SST_{\mathcal{B}}(\mu) \times SST_{\mathcal{B}}(\nu) \times SST_{\mathcal{A}}(\nu) \quad \text{by (2.3)} \\
& \xleftarrow{1-1} \bigsqcup_{\substack{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}} \\ \mu \subset \lambda}} SST_{\mathcal{B}}(\lambda) \times \left(\bigsqcup_{\nu \in \mathcal{P}_{\mathcal{A}}} SST_{\mathcal{A}}(\nu) \times \mathbf{LR}_{\nu \mu}^{\lambda} \right) \quad \text{by (2.1)} \\
& \xleftarrow{1-1} \bigsqcup_{\substack{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}} \\ \mu \subset \lambda}} SST_{\mathcal{B}}(\lambda) \times SST_{\mathcal{A}}(\lambda/\mu) \quad \text{by (2.2)}
\end{aligned}$$

Hence we obtain a bijection

$$(2.5) \quad SST_{\mathcal{B}}(\mu) \times \mathcal{M}_{\mathcal{A}, \mathcal{B}} \longrightarrow \bigsqcup_{\substack{\lambda \in \mathcal{P}_{\mathcal{A}} \cap \mathcal{P}_{\mathcal{B}} \\ \mu \subset \lambda}} SST_{\mathcal{B}}(\lambda) \times SST_{\mathcal{A}}(\lambda/\mu).$$

3. RATIONAL SEMISTANDARD TABLEAUX

3.1. Rational semistandard tableaux for $\mathfrak{gl}_{>0}$. For convenience, we let for a skew Young diagram λ/μ ,

$$\mathcal{B}_{\lambda/\mu} = SST_{\mathbb{N}}(\lambda/\mu), \quad \mathcal{B}_{\lambda/\mu}^{\vee} = SST_{\mathbb{N}^{\vee}}((\lambda/\mu)^{\vee}).$$

Definition 3.1. For $\mu, \nu \in \mathcal{P}$, we define $\mathcal{B}_{\mu, \nu}$ to be the set of bitableaux (S, T) such that

- (1) $(S, T) \in \mathcal{B}_{\mu} \times \mathcal{B}_{\nu}^{\vee}$,
- (2) $\left| \{ i \mid S(i, 1) \leq k \} \right| + \left| \{ i \mid T(i, 1) \geq k^{\vee} \} \right| \leq k$ for $k \geq 1$.

For convenience, we identify $\mathcal{B}_{\mu, \emptyset}$ and $\mathcal{B}_{\emptyset, \nu}$ with \mathcal{B}_{μ} and \mathcal{B}_{ν}^{\vee} , respectively.

Example 3.2.

$$\left(\begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 & \\ 4 & & \end{array}, \begin{array}{cc} 7^{\vee} & \\ 5^{\vee} & 5^{\vee} \\ 4^{\vee} & 3^{\vee} \end{array} \right) \in \mathcal{B}_{(3,2,1), (2,2,1)}$$

Remark 3.3. Let $\mathfrak{gl}_{>0}$ be the general linear Lie algebra spanned by $\mathbb{N} \times \mathbb{N}$ complex matrices of finite support. Then $\mathcal{B}_{\mu, \nu}$ parameterizes a basis of an extremal weight module over the quantum group $U_q(\mathfrak{gl}_{>0})$ [10]. Recall that $\mathcal{B}_{\mu, \nu} \cap \left(SST_{[n]}(\mu) \times SST_{[n]^{\vee}}(\nu^{\vee}) \right)$ ($n \geq 2$) parameterizes a basis of a finite dimensional complex irreducible representation of \mathfrak{gl}_n [17].

Let us review an insertion algorithm for $\mathcal{B}_{\mu,\nu}$ [10], which is an infinite analogue of those for rational semistandard tableaux for \mathfrak{gl}_n [17, 18]. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu,\nu}$, we define $a \rightarrow (S, T)$ in the following way;

Suppose first that S is empty and T is a single column tableau. Let (T', a') be the pair obtained as follows.

- (1) If T contains $a^\vee, (a+1)^\vee, \dots, (b-1)^\vee$ but not b^\vee , then T' is the tableau obtained from T by replacing $a^\vee, (a+1)^\vee, \dots, (b-1)^\vee$ with $(a+1)^\vee, (a+2)^\vee, \dots, b^\vee$, and put $a' = b$.
- (2) If T does not contain a^\vee , then leave T unchanged and put $a' = a$.

Now, we suppose that S and T are arbitrary.

- (1) Apply the above process to the leftmost column of T with a .
- (2) Repeat (1) with a' and the next column to the right.
- (3) Continue this process to the right-most column of T to get a tableau T' and a'' .
- (4) Define

$$(a \rightarrow (S, T)) = ((a'' \rightarrow S), T').$$

Then $a \rightarrow (S, T) \in \mathcal{B}_{\sigma,\nu}$ for some $\sigma \in \mathcal{P}$ with $|\sigma/\mu| = 1$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{N}}$, we let $(w \rightarrow (S, T)) = (w_r \rightarrow (\dots (w_1 \rightarrow (S, T)) \dots))$.

Next, we define $(S, T) \leftarrow a^\vee$ to be the pair (S', T') obtained in the following way;

- (1) If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (2) in Definition 3.1, then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.
- (2) Otherwise, choose the smallest k such that a_k is bumped out of the k -th row in the row insertion of a into T^\vee and the insertion of a_k into the $(k+1)$ -st row violates the condition (2) in Definition 3.1.
- (2-a) Stop the row insertion of a into T^\vee when a_k is bumped out and let T' be the resulting tableau after taking \vee .
- (2-b) Remove a_k in the left-most column of S , which necessarily exists, and then play the jeu de taquin (see for example [6, Section 1.2]) to obtain a tableau S' .

In this case, $(S, T) \leftarrow a^\vee \in \mathcal{B}_{\sigma,\tau}$, where either (1) $|\mu/\sigma| = 1$ and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau/\nu| = 1$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{N}^\vee}$, we let $((S, T) \leftarrow w) = ((\dots ((S, T) \leftarrow w_1) \dots) \leftarrow w_r)$.

3.2. Non-commutative Littlewood-Richardson rule. Let us recall the Littlewood-Richardson rule for $\mathcal{B}_{\mu,\nu}$ (see [10, Proposition 4.9] for more details).

Suppose that $\mu, \nu \in \mathcal{P}$ are given. For $(S, T) \in \mathcal{B}_\nu^\vee \times \mathcal{B}_\mu$, consider $(w_{\text{col}}(T) \rightarrow (\emptyset, S))$. Suppose that $w_{\text{col}}(T) = w_1 \dots w_r$ and $(w_1 \dots w_k \rightarrow (\emptyset, S)) \in \mathcal{B}_{\mu^{(k)}, \nu}$ for

$1 \leq k \leq r$ with $\mu^{(r)} = \mu$. Let $(i_k, j_k) \in \mu$ correspond to w_k in T ($1 \leq k \leq r$). Then $\mu^{(k)}$ is obtained from $\mu^{(k-1)}$ by adding a box in the i_k -th row. Hence, the map sending (S, T) to $(w_{\text{col}}(T) \rightarrow (\emptyset, S))$ gives a bijection [10, Corollary 4.11]

$$(3.1) \quad \mathcal{B}_\nu^\vee \times \mathcal{B}_\mu \longrightarrow \mathcal{B}_{\mu, \nu}.$$

Next, let $(S, T) \in \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee$ be given. Consider $((S, \emptyset) \leftarrow w_{\text{col}}(T))$. Suppose that $w_{\text{col}}(T) = w_1 \dots w_r$ and $((S, \emptyset) \leftarrow w_1 \dots w_k) \in \mathcal{B}_{\mu^{(k)}, \nu^{(k)}}$ for $1 \leq k \leq r$. Let $(i_k, j_k) \in \nu$ correspond to w_k in T ($1 \leq k \leq r$). Define U to be the tableau of shape ν such that for $1 \leq k \leq r$

$$U(i_k, j_k) = \begin{cases} i, & \text{if } \mu^{(k)} \text{ is obtained from } \mu^{(k-1)} \text{ by removing a box in the } i\text{-th row,} \\ -j, & \text{if } \nu^{(k)} \text{ is obtained from } \nu^{(k-1)} \text{ by adding a box in the } j\text{-th row.} \end{cases}$$

Then $U = U_+ * U_-$, where $U_+ \in SST_{\mathbb{N}}(\lambda)$ and $U_- \in SST_{-\mathbb{N}}(\nu/\lambda)$ for some $\lambda \subset \nu$. Let $\sigma = \mu^{(r)}$ and $\tau = \nu^{(r)}$. We have $\iota(U_+) \in \mathbf{LR}_{\sigma\lambda}^\mu$ and $U_- \in \overline{\mathbf{LR}}_{\lambda\tau}^\nu$ for some λ , hence $j(U_-)_R \in \mathbf{LR}_{\tau\lambda}^\nu$ (see Section 2.2, 2.3). Therefore, we have a bijection [11, Proposition 4.3]

$$(3.2) \quad \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee \longrightarrow \bigsqcup_{\lambda, \sigma, \tau} \mathcal{B}_{\sigma, \tau} \times \mathbf{LR}_{\sigma\lambda}^\mu \times \mathbf{LR}_{\tau\lambda}^\nu,$$

where (S, T) is mapped to $((S, \emptyset) \leftarrow w_{\text{col}}(T)), \iota(U_+), j(U_-)_R$.

Now, we have

$$\begin{aligned} & \bigsqcup_{\lambda, \sigma, \tau} \mathcal{B}_{\sigma, \tau} \times \mathbf{LR}_{\sigma\lambda}^\mu \times \mathbf{LR}_{\tau\lambda}^\nu \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda, \sigma, \tau} \mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma \times \mathbf{LR}_{\sigma\lambda}^\mu \times \mathbf{LR}_{\tau\lambda}^\nu \quad \text{by (3.1)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda, \sigma, \tau} \mathcal{B}_\tau^\vee \times \mathbf{LR}_{\tau\lambda}^\nu \times \mathcal{B}_\sigma \times \mathbf{LR}_{\sigma\lambda}^\mu \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda \subset \mu, \nu} \mathcal{B}_{\nu/\lambda}^\vee \times \mathcal{B}_{\mu/\lambda} \quad \text{by (2.2).} \end{aligned}$$

Hence, we obtain the following bijection [11, Proposition 5.1]

$$(3.3) \quad \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee \longrightarrow \bigsqcup_{\lambda \subset \mu, \nu} \mathcal{B}_{\nu/\lambda}^\vee \times \mathcal{B}_{\mu/\lambda}.$$

4. PLACTIC ALGEBRA

4.1. A plactic algebra for $\mathfrak{gl}_{>0}$. Let t be an indeterminate. Define $\mathcal{W}(t)$ to be an associative $\mathbb{Q}[t, t^{-1}]$ -algebra with unity generated by u_i and u_{i^\vee} ($i \in \mathbb{N}$) subject to the following relations;

$$(4.1) \quad \begin{aligned} u_i u_j u_k &= u_i u_k u_j, & u_{k^\vee} u_{j^\vee} u_{i^\vee} &= u_{j^\vee} u_{k^\vee} u_{i^\vee} \quad (j \leq i < k), \\ u_i u_j u_k &= u_j u_i u_k, & u_{k^\vee} u_{j^\vee} u_{i^\vee} &= u_{k^\vee} u_{i^\vee} u_{j^\vee} \quad (j < i \leq k), \\ u_{i+1} u_{(i+1)^\vee} &= u_{i^\vee} u_i \quad (i \geq 1), & u_1 u_{1^\vee} &= t, \\ u_i u_{j^\vee} &= u_{j^\vee} u_i \quad (i \neq j). \end{aligned}$$

Let $\mathcal{W}(t)_+$ (resp. $\mathcal{W}(t)_-$) be the subalgebra of $\mathcal{W}(t)$ generated by u_i (resp. u_{i^\vee}) for $i \geq 1$. Then $\mathcal{W}(t)_\pm$ is isomorphic to the usual plactic algebra for $\mathfrak{gl}_{>0}$ over $\mathbb{Q}[t, t^{-1}]$ [13], where the first two relations in (4.1) are Knuth relations.

Let \mathcal{W} be the set of finite words with letters in $\mathbb{N} \cup \mathbb{N}^\vee$. For $w = w_1 \cdots w_r \in \mathcal{W}$, put $u_w = u_{w_1} \cdots u_{w_r} \in \mathcal{W}(t)$. It is well-known that if $w \in \mathcal{W}_\mathbb{N}$ (resp. $\mathcal{W}_{\mathbb{N}^\vee}$), then there exists a unique $T \in \mathcal{B}_\mu$ (resp. \mathcal{B}_μ^\vee) for some $\mu \in \mathcal{P}$ such that $u_w = u_{w_{\text{col}}(T)} = u_{w_{\text{row}}(T)}$.

For a skew Young diagram λ/μ and $T \in \mathcal{B}_{\lambda/\mu}$ or $\mathcal{B}_{\lambda/\mu}^\vee$, we let $u_T = u_{w_{\text{col}}(T)}$, and for $\mu, \nu \in \mathcal{P}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, we let $u_{(S, T)} = u_S u_T$.

Lemma 4.1. *For $p, q \geq 1$, let $S \in \mathcal{B}_{(1^p)}$ and $T \in \mathcal{B}_{(1^q)}^\vee$ be given and let $(S', T') = (w_{\text{col}}(S) \rightarrow T) \in \mathcal{B}_{(1^p), (1^q)}$. Then $u_T u_S = u_{S'} u_{T'}$.*

Proof. It is straightforward to check from (3.1) and (4.1). \square

Lemma 4.2. *For $p, q \geq 1$, let $S \in \mathcal{B}_{(1^p)}$ and $T \in \mathcal{B}_{(1^q)}^\vee$ be given with $w_{\text{col}}(S) = w_1^+ \cdots w_p^+$ and $w_{\text{col}}(T) = w_q^- \cdots w_1^-$. Suppose that there exists $k \geq 1$ such that $|\{i \mid w_i^+ \leq k\}| + |\{j \mid w_j^- \geq k^\vee\}| > k$. If $w_i^+ = k$ and $w_j^- = k^\vee$ for some i and j , and $(S, T') \in \mathcal{B}_{(1^p), (1^{q-1})}$, where T' is obtained from T by removing k^\vee , then*

$$u_S u_T = t u_{w_1^+ \cdots \widehat{w_i^+} \cdots w_p^+} u_{w_q^- \cdots \widehat{w_j^-} \cdots w_1^-}.$$

Proof. We use induction on $p+q$. If $p+q = 2$, then $k = 1$ and $u_{w_1^+} u_{w_1^-} = u_1 u_{1^\vee} = t$.

Suppose that $p+q \geq 2$ with $i < p$ and $j < q$. Note that

$$\begin{aligned} w_{i+1}^+ \cdots w_p^+ &= (k + a_1) \cdots (k + a_{p-i}), \\ w_q^- \cdots w_{j+1}^- &= (k + b_{q-j})^\vee \cdots (k + b_1)^\vee, \end{aligned}$$

for some $a_1 < \cdots < a_{p-i}$ and $b_1 < \cdots < b_{q-j}$. Also it follows from our hypothesis that $i + j = k + 1$, and hence we can choose $(\overline{S}, \overline{T}) \in \mathcal{B}_{(1^{p-i}), (1^{q-j})}$ such that $w_{\text{col}}(\overline{S}) = a_1 \cdots a_{p-i}$ and $w_{\text{col}}(\overline{T}) = b_{q-j}^\vee \cdots b_1^\vee$. By Lemma 4.1, there exists $(\overline{T}', \overline{S}') \in$

$\mathcal{B}_{(1^{q-j})}^\vee \times \mathcal{B}_{(1^{p-i})}$ with $w_{\text{col}}(\overline{S}') = a'_1 \dots a'_{p-i}$ and $w_{\text{col}}(\overline{T}') = (b'_{q-j})^\vee \dots (b'_1)^\vee$ such that $u_{\overline{T}'} u_{\overline{S}'} = u_{\overline{S}} u_{\overline{T}}$. This implies that

$$u_{w_{i+1}^+ \dots w_p^+} u_{w_q^- \dots w_{j+1}^-} = u_{(k+b'_{q-j})^\vee \dots u_{(k+b'_1)^\vee} u_{(k+a'_1)} \dots u_{(k+a'_{p-i})}$$

Since $w_i^+ = k < k + b'_1$ and $w_j^- = k^\vee > (k + a'_1)^\vee$, we have

$$u_S u_T = u_{(k+b'_{q-j})^\vee \dots (k+b'_1)^\vee} \left(u_{w_1^+} \dots u_{w_i^+} u_{w_j^-} \dots u_{w_1^-} \right) u_{(k+a'_1) \dots (k+a'_{p-i})},$$

and by induction hypothesis,

$$\begin{aligned} u_S u_T &= t u_{(k+b'_{q-j})^\vee \dots (k+b'_1)^\vee} \left(u_{w_1^+} \dots \widehat{u_{w_i^+}} \widehat{u_{w_j^-}} \dots u_{w_1^-} \right) u_{(k+a'_1) \dots (k+a'_{p-i})} \\ &= t u_{w_1^+} \dots \widehat{u_{w_i^+}} \dots u_{w_p^+} u_{w_q^-} \dots \widehat{u_{w_j^-}} \dots u_{w_1^-}. \end{aligned}$$

Now, we assume that $i = p$ and $j = q$, that is, $w_p^+ = k$ and $w_q^- = k^\vee$. Note that $p + q = k + 1$. If $w_{p-1}^+ \neq k - 1$ and $w_{q-1}^- \neq (k - 1)^\vee$, then

$$\left| \{i \mid w_i^+ \leq k - 2\} \right| + \left| \{j \mid w_j^- \geq (k - 2)^\vee\} \right| = p + q - 2 = k - 1 > k - 2,$$

which contradicts the fact that $(S, T') \in \mathcal{B}_{(1^p), (1^{q-1})}$. So we also assume that either $w_{p-1}^+ = k - 1$ or $w_{q-1}^- = (k - 1)^\vee$.

CASE 1. Suppose that $w_{p-1}^+ \neq k - 1$ and $w_{q-1}^- = (k - 1)^\vee$. We have

$$\begin{aligned} u_S u_T &= u_{w_1^+} \dots u_{w_{p-1}^+} u_k u_{k^\vee} u_{w_{q-1}^-} \dots u_{w_1^-} \\ &= u_{w_1^+} \dots u_{w_{p-1}^+} u_{(k-1)^\vee} u_{k-1} u_{w_{q-1}^-} \dots u_{w_1^-} \\ &= u_{(k-1)^\vee} u_{w_1^+} \dots u_{w_{p-1}^+} u_{k-1} u_{(k-1)^\vee} u_{w_{q-2}^-} \dots u_{w_1^-} \\ &= t u_{(k-1)^\vee} u_{w_1^+} \dots u_{w_{p-1}^+} u_{w_{q-2}^-} \dots u_{w_1^-} \\ &= t u_{w_1^+} \dots u_{w_{p-1}^+} u_{(k-1)^\vee} u_{w_{q-2}^-} \dots u_{w_1^-} \\ &= t u_{w_1^+} \dots u_{w_{p-1}^+} u_{w_{q-1}^-} \dots u_{w_1^-}, \end{aligned}$$

where we use induction hypothesis in the third line.

CASE 2. Suppose that $w_{p-1}^+ = k - 1$ and $w_{q-1}^- \neq (k - 1)^\vee$. By almost the same argument as in CASE 1, we have

$$u_S u_T = t u_{w_1^+} \dots u_{w_{p-1}^+} u_{w_{q-1}^-} \dots u_{w_1^-}.$$

CASE 3. Suppose that $w_{p-1}^+ = k - 1$ and $w_{q-1}^- = (k - 1)^\vee$. We have

$$\begin{aligned} u_S u_T &= u_{w_1^+} \dots u_{w_{p-1}^+} u_k u_{k^\vee} u_{w_{q-1}^-} \dots u_{w_1^-} \\ &= u_{w_1^+} \dots u_{w_{p-1}^+} u_{(k-1)^\vee} u_{k-1} u_{w_{q-1}^-} \dots u_{w_1^-} \\ &= u_{(k-a)^\vee} u_{v_1} \dots u_{v_{p-2}} u_{k-2} u_{k-1} u_{w_{q-1}^-} \dots u_{w_1^-}, \end{aligned}$$

for some $1 \leq a < k$ and $1 \leq v_1 < \dots < v_{p-2} \leq k-2$. So by induction hypothesis,

$$\begin{aligned}
u_S u_T &= u_{(k-a)^\vee} u_{v_1} \dots u_{v_{p-2}} u_{k-2} u_{k-1} u_{(k-1)^\vee} u_{w_{q-2}^-} \dots u_{w_1^-} \\
&= t u_{(k-a)^\vee} u_{v_1} \dots u_{v_{p-2}} u_{k-2} u_{w_{q-2}^-} \dots u_{w_1^-} \\
&= t u_{w_1^+} \dots u_{w_{p-2}^+} u_{k-1} u_{(k-1)^\vee} u_{w_{q-1}^-} \dots u_{w_1^-} \\
&= t u_{w_1^+} \dots u_{w_{p-1}^+} u_{w_{q-1}^-} \dots u_{w_1^-}.
\end{aligned}$$

This completes the induction. \square

Lemma 4.3. *Let $\mu, \nu \in \mathcal{P}$ be given. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$,*

- (1) $u_{(S, T)} u_a = u_{(a \rightarrow (S, T))}$,
- (2) $u_{(S, T)} u_{a^\vee} = t^\epsilon u_{((S, T) \leftarrow a^\vee)}$, where $\epsilon = 0, 1$.

Proof. We keep the notations in Section 3.1. Consider $(a \rightarrow (S, T)) = (S', T')$. Let (T', a') be the pair obtained by the first step in the definition of $a \rightarrow (S, T)$. It is straightforward to check that $u_T u_a = u_{a'} u_{T'}$. Since $S' = (a' \rightarrow S)$, which is a usual column insertion, we have $u_{S'} = u_S u_{a'}$. Hence

$$u_{(S, T)} u_a = u_S u_T u_a = u_S u_{a'} u_{T'} = u_{S'} u_{T'} = u_{(a \rightarrow (S, T))}.$$

Next, consider $((S, T) \leftarrow a^\vee) = (S', T')$. If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (2) in Definition 3.1, then $(S', T') = (S, (T^\vee \leftarrow a)^\vee)$, which implies that $u_S = u_{S'}$ and $u_T u_{a^\vee} = u_{T'}$. Hence, $u_{(S, T)} u_{a^\vee} = u_{((S, T) \leftarrow a^\vee)}$.

Suppose that there exists j such that $a_j = k$ is bumped out of the $(j-1)$ -st row in the row insertion of a into T^\vee and the insertion of a_j into the j -th row violates the condition (2) in Definition 3.1.

Let $T'' = (T^\vee \leftarrow a)^\vee$. Suppose that $w_{\text{col}}(S) = \tilde{w}^+ w^+$ and $w_{\text{col}}(T'') = w^- \tilde{w}^-$, where $w^+ = w_1^+ \dots w_p^+$ is the subword corresponding to the leftmost column of S and $w^- = w_q^- \dots w_1^-$ is the subword corresponding to the rightmost column of T'' reading from top to bottom. Note that $w_j^- = k^\vee$. Suppose that $w_i^+ = k$. By Lemma 4.2, we have

$$u_{w^+} u_{w^-} = t u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+} u_{w_q^- \dots \widehat{w_j^-} \dots w_1^-}.$$

Note that $w_{\text{col}}(T') = w_q^- \dots \widehat{w_j^-} \dots w_1^- \tilde{w}^-$. Recalling that S' is obtained by playing the jeu de taquin after removing k in the first column of S , it follows that $u_{S'} = u_{\tilde{w}^+} u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+}$. Therefore,

$$\begin{aligned}
u_{(S, T)} u_{a^\vee} &= u_S u_T u_{a^\vee} = u_S u_{T''} \\
&= u_{\tilde{w}^+} u_{w^+} u_{w^-} u_{\tilde{w}^-} \\
&= t u_{\tilde{w}^+} u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+} u_{w_q^- \dots \widehat{w_j^-} \dots w_1^-} u_{\tilde{w}^-}
\end{aligned}$$

$$= t u_{S'} u_{T'} = t u_{((S,T) \leftarrow a^\vee)}.$$

□

Now, we obtain the following immediately.

Proposition 4.4. *For $w = w_1 \dots w_r \in \mathcal{W}$, there exists $(S, T) \in \mathcal{B}_{\mu, \nu}$ such that $u_w = t^\epsilon u_{(S, T)}$ where $\epsilon = r - |\mu| - |\nu|$.*

Corollary 4.5. *The set $\{u_{(S, T)} \mid (S, T) \in \mathcal{B}_{\mu, \nu}, \mu, \nu \in \mathcal{P}\}$ spans $\mathcal{U}(t)$ over $\mathbb{Q}[t, t^{-1}]$.*

The uniqueness of (S, T) in Proposition 4.4 and the linear independence of the spanning set in Corollary 4.5 will be proved in Section 4.2.

4.2. Crystal equivalence. Let $P = P_{\mathbb{N}} = \bigoplus_{i \geq 1} \mathbb{Z} \epsilon_i$ be the weight lattice of $\mathfrak{gl}_{>0}$ and $\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \geq 1\}$ the set of simple roots of $\mathfrak{gl}_{>0}$. A (regular) $\mathfrak{gl}_{>0}$ -crystal is a set B together with the maps $\text{wt} : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}_{\geq 0}$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{\mathbf{0}\}$ ($i \geq 1$) such that for $b \in B$ and $i \geq 1$

- (1) $\varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b)$,
- (2) $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq \mathbf{0}\}$
- (3) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq \mathbf{0}$, and $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq \mathbf{0}$,
- (4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,

where $\mathbf{0}$ is a formal symbol (cf. [7]). Note that B is equipped with a colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \geq 1$. The dual crystal B^\vee of B is defined to be the set $\{b^\vee \mid b \in B\}$ with $\text{wt}(b^\vee) = -\text{wt}(b)$, $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ and $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$ for $b \in B$ and $i \geq 1$. We assume that $\mathbf{0}^\vee = \mathbf{0}$. Note that \mathbb{N} is naturally equipped with a $\mathfrak{gl}_{>0}$ -crystal structure;

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \dots$$

with $\text{wt}(i) = \epsilon_i$ ($i \geq 1$), while \mathbb{N}^\vee is its dual.

For crystals B_1 and B_2 , a tensor product $B_1 \otimes B_2$ is defined to be the set $\{b_1 \otimes b_2 \mid b_i \in B_i \text{ } (i = 1, 2)\}$ with $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$, and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

for $i \geq 1$ and $b_1 \otimes b_2 \in B_1 \otimes B_2$, where we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. For example, \mathcal{W} is a $\mathfrak{gl}_{>0}$ -crystal, where each word $w_1 \dots w_r$ is identified with $w_1 \otimes \dots \otimes w_r$ in a mixed r -tensor product of \mathbb{N} and \mathbb{N}^\vee .

For $b_i \in B_i$ ($i = 1, 2$), we say that b_1 is equivalent to b_2 , and write $b_1 \equiv b_2$ if $\text{wt}(b_1) = \text{wt}(b_2)$ and they generate the same \mathbb{N} -colored graph with respect to \tilde{e}_i, \tilde{f}_i ($i \geq 1$). We usually call \equiv the crystal equivalence..

For a skew Young diagram λ/μ , $\mathcal{B}_{\lambda/\mu}$ has a well-defined $\mathfrak{gl}_{>0}$ -crystal structure such that $\tilde{x}_i(S) = S'$ if $\tilde{x}_i w_{\text{col}}(S) \neq \mathbf{0}$ ($i \geq 1, x = e, f$), where S' is the unique tableau in $\mathcal{B}_{\lambda/\mu}$ with $w_{\text{col}}(S') = \tilde{x}_i w_{\text{col}}(S)$, and $\tilde{x}_i(S) = \mathbf{0}$ otherwise. We regard $\mathcal{B}_{\lambda/\mu}^\vee$ as the dual of $\mathcal{B}_{\lambda/\mu}$. Moreover, for $\mu, \nu \in \mathcal{P}$, $\mathcal{B}_{\mu, \nu} \cup \{\mathbf{0}\} \subset (\mathcal{B}_\mu \otimes \mathcal{B}_\nu^\vee) \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i, \tilde{f}_i ($i \geq 1$), and hence a $\mathfrak{gl}_{>0}$ -crystal, which is connected as a graph [10, Proposition 3.4].

Let \mathcal{W} be an associative \mathbb{Q} -algebra generated by the symbol $[w]$ ($w \in \mathcal{W}$) subject to the relations;

$$\begin{aligned} [w][w'] &= [ww'], \\ [w] &= [w'], \quad \text{if } w \equiv w', \end{aligned}$$

for $w, w' \in \mathcal{W}$. Note that $[\emptyset] = 1$ is the unity in \mathcal{W} , where \emptyset is the empty word.

Lemma 4.6. *The set*

$$\mathcal{B} = \{ [w_{\text{col}}(S)w_{\text{col}}(T)] \mid (S, T) \in \mathcal{B}_{\mu, \nu}, \mu, \nu \in \mathcal{P} \}$$

is a \mathbb{Q} -basis of \mathcal{W} .

Proof. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu, \nu}$, it is shown in [10] that

$$(a \rightarrow (S, T)) \equiv (S, T) \otimes a, \quad ((S, T) \leftarrow a^\vee) \equiv (S, T) \otimes a^\vee.$$

This implies that for $w \in \mathcal{W}$, $[w] = [w_{\text{col}}(S)w_{\text{col}}(T)]$ for some $(S, T) \in \mathcal{B}_{\mu, \nu}$, and hence \mathcal{W} is spanned by \mathcal{B} .

Now, suppose that

$$(4.2) \quad \sum_{i=1}^n c_i [w_{\text{col}}(S^{(i)})w_{\text{col}}(T^{(i)})] = 0$$

for some $c_i \in \mathbb{Q}$ and $(S^{(i)}, T^{(i)}) \in \mathcal{B}_{\mu^{(i)}, \nu^{(i)}}$ with $\mu^{(i)}, \nu^{(i)} \in \mathcal{P}$ ($1 \leq i \leq n$). Since $(S, T) \equiv (S', T')$ implies $(S, T) = (S', T')$ for $(S, T) \in \mathcal{B}_{\mu, \nu}$ and $(S', T') \in \mathcal{B}_{\sigma, \tau}$ [10, Lemma 5.1], we assume that $(S^{(i)}, T^{(i)})$'s are mutually different.

We use induction on n to show that $c_i = 0$ for $1 \leq i \leq n$. It is clear when $n = 1$. Suppose that $n \geq 2$. Since no two of $(S^{(i)}, T^{(i)})$'s are mutually crystal equivalent, there exist j_1, \dots, j_r such that $\tilde{x}_{j_1} \cdots \tilde{x}_{j_r}(S^{(1)}, T^{(1)}) = \mathbf{0}$ but $\tilde{x}_{j_1} \cdots \tilde{x}_{j_r}(S^{(i)}, T^{(i)}) \neq \mathbf{0}$ for some $2 \leq i \leq n$, where $x = e$ or f for each j_k ($1 \leq k \leq r$).

Note that \tilde{x}_i ($x = e, f, i \geq 1$) acts on \mathscr{W} by $\tilde{x}_i[w] = [\tilde{x}_i w]$, where we assume that $[\mathbf{0}] = 0$. Hence by applying $X = \tilde{x}_{j_1} \cdots \tilde{x}_{j_r}$ to (4.2), we get

$$\sum_{i=2}^n c_i \left[X w_{\text{col}}(S^{(i)}) w_{\text{col}}(T^{(i)}) \right] = \sum_{i=2}^n c_i \left[w_{\text{col}}(\overline{S}^{(i)}) w_{\text{col}}(\overline{T}^{(i)}) \right] = 0$$

for some $[w_{\text{col}}(\overline{S}^{(i)}) w_{\text{col}}(\overline{T}^{(i)})] \in \mathcal{B}$.

Here, we assume that $c_i = 0$ if $X(w_{\text{col}}(S^{(i)}) w_{\text{col}}(T^{(i)})) = \mathbf{0}$. By induction hypothesis, we have $c_2 = \cdots = c_n = 0$, and hence $c_1 = 0$. Therefore, \mathcal{B} is a \mathbb{Q} -basis of \mathscr{W} . \square

Theorem 4.7. *Let $\mathscr{U}(1)$ be the \mathbb{Q} -algebra obtained from $\mathscr{U}(t)$ by specializing $t = 1$. Then as a \mathbb{Q} -algebra, we have*

$$\mathscr{U}(1) \simeq \mathscr{W},$$

where u_a is mapped to $[a]$ for $a \in \mathbb{N} \cup \mathbb{N}^\vee$.

Proof. It is straightforward to see that the relations in (4.4) are satisfied in \mathscr{W} under the correspondence $u_a \mapsto [a]$. Hence there exists a \mathbb{Q} -algebra homomorphism $\psi : \mathscr{U}(1) \rightarrow \mathscr{W}$ sending u_a to $[a]$. Since $\{u_{(S,T)} \mid (S,T) \in \mathcal{B}_{\mu,\nu}, \mu, \nu \in \mathcal{P}\}$ spans $\mathscr{U}(1)$ and $\psi(u_{(S,T)}) = [w_{\text{col}}(S) w_{\text{col}}(T)]$, it follows from Lemma 4.6 that ψ is an isomorphism. \square

Corollary 4.8. *The set*

$$\{u_{(S,T)} \mid (S,T) \in \mathcal{B}_{\mu,\nu}, \mu, \nu \in \mathcal{P}\}$$

is a $\mathbb{Q}[t, t^{-1}]$ -basis of $\mathscr{U}(t)$.

Proof. Since $\{u_{(S,T)} \mid (S,T) \in \mathcal{B}_{\mu,\nu}, \mu, \nu \in \mathcal{P}\} \subset \mathscr{U}(1)$ is mapped to \mathcal{B} by Theorem 4.7, it is a \mathbb{Q} -basis of $\mathscr{U}(1)$. Now there is a well-defined \mathbb{Q} -algebra homomorphism $\psi' : \mathscr{U}(t) \rightarrow \mathscr{U}(1)$ such that $\psi'(u_a) = u_a$ and $\psi'(t) = 1$. Then it is not difficult to check that $\{u_{(S,T)} \mid (S,T) \in \mathcal{B}_{\mu,\nu}, \mu, \nu \in \mathcal{P}\} \subset \mathscr{U}(t)$ is linearly independent over $\mathbb{Q}[t, t^{-1}]$ and hence a $\mathbb{Q}[t, t^{-1}]$ -basis of $\mathscr{U}(t)$. \square

Corollary 4.9. *For $w \in \mathcal{W}$, there exists unique $(S,T) \in \mathcal{B}_{\mu,\nu}$ and $\epsilon \in \mathbb{Z}_{\geq 0}$ such that $u_w = t^\epsilon u_{(S,T)}$.*

4.3. Non-commutative Schur functions. Let $\widehat{\mathscr{U}(t)} = \bigoplus_{n \geq 0} \widehat{\mathscr{U}(t)}_n$, where $\widehat{\mathscr{U}(t)}_n$ is the completion of $\bigoplus_{|\mu|+|\nu|=n} \bigoplus_{(S,T) \in \mathcal{B}_{\mu,\nu}} \mathbb{Q}[t, t^{-1}] u_{(S,T)}$. For a skew Young diagram λ/μ , let

$$s_{\lambda/\mu}(\mathbf{u}) = \sum_{S \in \mathcal{B}_{\lambda/\mu}} u_S, \quad s_{\lambda/\mu}^\vee(\mathbf{u}) = \sum_{S \in \mathcal{B}_{\lambda/\mu}^\vee} u_S \in \widehat{\mathscr{U}(t)},$$

which are plactic skew Schur functions in u_i 's and u_{i^\vee} 's, respectively.

Let $\Lambda(t)$ be the algebra of symmetric functions in $\mathbf{x} = \mathbf{x}_{\mathbb{N}}$ over $\mathbb{Q}[t, t^{-1}]$. Then $\{s_{(k)}(\mathbf{u}) \mid k \geq 1\}$ (resp. $\{s_{(k)}^{\vee}(\mathbf{u}) \mid k \geq 1\}$) generates the subalgebra $\mathcal{S}(t)_{\pm}$ of $\widehat{\mathcal{U}(t)}$ isomorphic to $\Lambda(t)$ [13], where $s_{(k)}(\mathbf{u})$ (resp. $s_{(k)}^{\vee}(\mathbf{u})$) corresponds to the k -th complete symmetric function $h_k(\mathbf{x}) = s_{(k)}(\mathbf{x})$, and $\{s_{\lambda}(\mathbf{u}) \mid \lambda \in \mathcal{P}\}$ (resp. $\{s_{\lambda}^{\vee}(\mathbf{u}) \mid \lambda \in \mathcal{P}\}$) is a $\mathbb{Q}[t, t^{-1}]$ -basis of $\mathcal{S}(t)_{+}$ (resp. $\mathcal{S}(t)_{-}$).

We define

$$s_{\mu, \nu}(\mathbf{u}) = \sum_{(S, T) \in \mathcal{B}_{\mu, \nu}} u_{(S, T)}$$

for $\mu, \nu \in \mathcal{P}$ and let

$$\mathcal{S}(t) = \sum_{\mu, \nu \in \mathcal{P}} \mathbb{Q}[t, t^{-1}] s_{\mu, \nu}(\mathbf{u}) \subset \widehat{\mathcal{U}(t)}.$$

Lemma 4.10. *For $\mu, \nu \in \mathcal{P}$, we have*

$$s_{\mu}(\mathbf{u}) s_{\nu}^{\vee}(\mathbf{u}) = \sum_{\lambda \subset \mu, \nu} t^{|\lambda|} s_{\nu/\lambda}^{\vee}(\mathbf{u}) s_{\mu/\lambda}(\mathbf{u}) = \sum_{\lambda, \sigma, \tau} t^{|\lambda|} c_{\lambda \sigma}^{\mu} c_{\lambda \tau}^{\nu} s_{\sigma, \tau}(\mathbf{u}).$$

Proof. By (3.1) and Lemma 4.3 (1), we have $s_{\mu, \nu}(\mathbf{u}) = s_{\nu}^{\vee}(\mathbf{u}) s_{\mu}(\mathbf{u})$. The identity follows from (3.3) and Lemma 4.3 (2). \square

Proposition 4.11. *$\mathcal{S}(t)$ is a $\mathbb{Q}[t, t^{-1}]$ -algebra with a basis $\{s_{\mu, \nu}(\mathbf{u}) \mid \mu, \nu \in \mathcal{P}\}$, where*

$$s_{\mu, \nu}(\mathbf{u}) s_{\sigma, \tau}(\mathbf{u}) = \sum_{\zeta, \eta} \left(\sum_{\alpha, \beta, \gamma} t^{|\gamma|} c_{\sigma \alpha}^{\zeta} c_{\alpha \beta}^{\mu} c_{\beta \gamma}^{\tau} c_{\gamma \nu}^{\eta} \right) s_{\zeta, \eta}(\mathbf{u}).$$

for $\mu, \nu, \sigma, \tau \in \mathcal{P}$.

Proof. In fact, $\{s_{\mu, \nu}(\mathbf{u}) \mid \mu, \nu \in \mathcal{P}\}$ is linearly independent by Lemma 4.8, and hence a basis of $\mathcal{S}(t)$. Combining Lemma 4.10 with the usual Littlewood-Richardson rule (2.1) for $s_{\mu}(\mathbf{u})$'s and $s_{\nu}^{\vee}(\mathbf{u})$'s, we obtain the above identity. Since the sum on the right hand side is finite, $\mathcal{S}(t)$ has a well-defined multiplication and hence is a $\mathbb{Q}[t, t^{-1}]$ -algebra. \square

4.4. Heisenberg algebra. Let $\mathcal{H}(t)$ be an associative $\mathbb{Q}[t, t^{-1}]$ -algebra with unity generated by B_n ($n \in \mathbb{Z} \setminus \{0\}$) subject to the relations;

$$B_k B_l - B_l B_k = k t^k \delta_{k+l, 0}.$$

For $k \geq 1$, let $p_k(\mathbf{u}) \in \mathcal{S}(t)_{+}$ (resp. $p_k^{\vee}(\mathbf{u}) \in \mathcal{S}(t)_{-}$) correspond to the k -th power sum symmetric function $p_k(\mathbf{x}) \in \Lambda(t)$.

Proposition 4.12. *As a $\mathbb{Q}[t, t^{-1}]$ -algebra, we have*

$$\mathcal{S}(t) \simeq \mathcal{H}(t),$$

where $p_k(\mathbf{u})$ and $p_k^{\vee}(\mathbf{u})$ are mapped to B_k and B_{-k} for $k \geq 1$, respectively.

Proof. By Lemma 4.10, we have

$$(4.3) \quad h_s(\mathbf{u})h_r^\vee(\mathbf{u}) = \sum_{i=0}^m t^i h_{r-i}^\vee(\mathbf{u})h_{s-i}(\mathbf{u})$$

for $r, s \geq 1$, where $m = \min\{r, s\}$. Here we assume that $h_0(\mathbf{u}) = h_0^\vee(\mathbf{u}) = 1$. We may view $\mathcal{S}(t)$ as an algebra generated by $\{h_s^\vee(\mathbf{u}), h_r(\mathbf{u}) \mid r, s \geq 1\}$ with the defining relations (4.3). Since

$$h_r(\mathbf{u}) = \sum_{|\lambda|=r} \frac{1}{z_\lambda} p_\lambda(\mathbf{u}),$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ and $m_i(\lambda) = |\{k \mid \lambda_k = i\}|$, we obtain

$$p_k(\mathbf{u})p_l^\vee(\mathbf{u}) - p_l^\vee(\mathbf{u})p_k(\mathbf{u}) = k t^k \delta_{k,l}$$

by using the same argument as in [12, Corollary 8]. This implies that the map $\psi : \mathcal{H}(t) \rightarrow \mathcal{S}(t)$ sending B_{-k} (resp. B_k) to $p_k^\vee(\mathbf{u})$ (resp. $p_k(\mathbf{u})$) for $k \geq 1$ is a well-defined isomorphism. \square

Remark 4.13. Regarding $\mathcal{S}(0)$ and $\mathcal{S}(1)$ as \mathbb{Q} -algebras generated by $h_k(\mathbf{u})$ and $h_k^\vee(\mathbf{u})$ ($k \geq 1$), we have $\mathcal{S}(0) \simeq \Lambda \otimes \Lambda$, and $\mathcal{S}(1) \simeq \langle \frac{\partial}{\partial p_k}, p_k \mid k \geq 1 \rangle \subset \text{End}_{\mathbb{Q}}(\Lambda)$, where Λ is the algebra of symmetric functions in \mathbf{x} over \mathbb{Q} and p_k is the operator on Λ induced from the multiplication by $p_k(\mathbf{x})$. Therefore, we may view $\mathcal{S}(t)$ as an algebra interpolating the algebra of double symmetric functions and the Weyl algebra of infinite rank.

5. KNUTH CORRESPONDENCE AND CAUCHY IDENTITY

5.1. Main result. Let \mathcal{A} and \mathcal{B} be linearly ordered \mathbb{Z}_2 -graded sets. For $A \in \mathcal{M}_{\mathcal{A}, \mathbb{N}}$ (or $\mathcal{M}_{\mathcal{A}, \mathbb{N}^\vee}$), we put $u_A = u_{\mathbf{j}}$ if $A = A(\mathbf{i}, \mathbf{j})$. Now we are in a position to state and prove our main theorem.

Theorem 5.1. *There exists a bijection*

$$\mathcal{M}_{\mathcal{A}, \mathbb{N}} \times \mathcal{M}_{\mathcal{B}, \mathbb{N}^\vee} \longrightarrow \mathcal{M}_{\mathcal{B}, \mathbb{N}^\vee} \times \mathcal{M}_{\mathcal{A}, \mathbb{N}} \times \mathcal{M}_{\mathcal{A}, \mathcal{B}}$$

sending (X, Y) to (Y', X', Z) such that

- (1) $\text{wt}_{\mathcal{A}}(X) = \text{wt}_{\mathcal{A}}(X') + \text{wt}_{\mathcal{A}}(Z)$, $\text{wt}_{\mathcal{B}}(Y) = \text{wt}_{\mathcal{B}}(Y') + \text{wt}_{\mathcal{B}}(Z)$,
- (2) $u_X u_Y = t^{|Z|} u_{Y'} u_{X'}$ where $Z = (z_{ij})$ and $|Z| = \sum_{i,j} z_{ij}$.

Proof. It is obtained by composing the following bijections;

$$\begin{aligned} & \mathcal{M}_{\mathcal{A}, \mathbb{N}} \times \mathcal{M}_{\mathcal{B}, \mathbb{N}^\vee} \\ & \longrightarrow \bigsqcup_{\mu \in \mathcal{P}_{\mathcal{A}}, \nu \in \mathcal{P}_{\mathcal{B}}} \mathcal{B}_\mu \times SST_{\mathcal{A}}(\mu) \times \mathcal{B}_\nu^\vee \times SST_{\mathcal{B}}(\nu) \end{aligned} \quad \text{by (2.3) and (2.4)}$$

$$\begin{aligned}
& \longrightarrow \bigsqcup_{\mu \in \mathcal{P}_A, \nu \in \mathcal{P}_B} SST_A(\mu) \times SST_B(\nu) \times \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee \\
& \longrightarrow \bigsqcup_{\mu \in \mathcal{P}_A, \nu \in \mathcal{P}_B} SST_A(\mu) \times SST_B(\nu) \times \left(\bigsqcup_{\lambda \subset \mu, \nu} \mathcal{B}_{\nu/\lambda}^\vee \times \mathcal{B}_{\mu/\lambda} \right) \quad \text{by (3.3)} \\
& \longrightarrow \bigsqcup_{\mu \in \mathcal{P}_A, \nu \in \mathcal{P}_B} \bigsqcup_{\lambda \subset \mu, \nu} SST_B(\nu) \times \mathcal{B}_{\nu/\lambda}^\vee \times SST_A(\mu) \times \mathcal{B}_{\mu/\lambda} \\
& \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_A \cap \mathcal{P}_B} \mathcal{M}_{B, \mathbb{N}^\vee} \times SST_B(\lambda) \times \mathcal{M}_{A, \mathbb{N}} \times SST_A(\lambda) \quad \text{by (2.5)} \\
& \longrightarrow \mathcal{M}_{B, \mathbb{N}^\vee} \times \mathcal{M}_{A, \mathbb{N}} \times \left(\bigsqcup_{\lambda \in \mathcal{P}_A \cap \mathcal{P}_B} SST_B(\lambda) \times SST_A(\lambda) \right) \\
& \longrightarrow \mathcal{M}_{B, \mathbb{N}^\vee} \times \mathcal{M}_{A, \mathbb{N}} \times \mathcal{M}_{A, \mathbb{B}} \quad \text{by (2.3).}
\end{aligned}$$

□

Now, let us consider the non-commutative character identity associated with Theorem 5.1. We first define the plactic Cauchy products

$$\mathcal{Q}(\mathbf{x}_A) = \overrightarrow{\prod_{a \in A}} \mathcal{Q}(x_a), \quad \mathcal{P}(\mathbf{x}_B) = \overrightarrow{\prod_{b \in B}} \mathcal{P}(x_b),$$

where the products are given with respect to the linear ordering on A or B so that smaller terms are to the left, and

$$\begin{aligned}
\mathcal{P}(x_b) &= \begin{cases} \frac{1}{\cdots (1 - u_2^\vee x_b)(1 - u_1^\vee x_b)}, & \text{if } |b| = 0, \\ (1 + u_1^\vee x_b)(1 + u_2^\vee x_b) \cdots, & \text{if } |b| = 1, \end{cases} \\
\mathcal{Q}(x_a) &= \begin{cases} \frac{1}{\cdots (1 - u_2 x_a)(1 - u_1 x_a)}, & \text{if } |a| = 0, \\ (1 + u_1 x_a)(1 + u_2 x_a) \cdots, & \text{if } |a| = 1. \end{cases}
\end{aligned}$$

We assume that \mathbf{x}_A and \mathbf{x}_B commute with \mathbf{u} . Note that

$$\mathcal{Q}(\mathbf{x}_A) = \sum_{\lambda \in \mathcal{P}_A} s_\lambda(\mathbf{u}) s_\lambda(\mathbf{x}_A), \quad \mathcal{P}(\mathbf{x}_B) = \sum_{\lambda \in \mathcal{P}_B} s_\lambda^\vee(\mathbf{u}) s_\lambda(\mathbf{x}_B),$$

by (2.3) and (2.4). Since the bijections in the proof of Theorem 5.1 preserves the plactic relations (4.1), wt_A and wt_B , we obtain the following identity.

Corollary 5.2.

$$\mathcal{Q}(\mathbf{x}_A) \mathcal{P}(\mathbf{x}_B) = \mathcal{P}(\mathbf{x}_B) \mathcal{Q}(\mathbf{x}_A) \frac{\prod_{|a| \neq |b|} (1 + tx_a x_b)}{\prod_{|a| = |b|} (1 - tx_a x_b)}.$$

5.2. Cauchy identity for Schur operators. For $i \in \mathbb{N}$, we define operators $\bar{u}_i, \bar{u}_{i^\vee} \in \text{End}_{\mathbb{Q}[t, t^{-1}]}(\Lambda(t))$ by

$$\begin{aligned}\bar{u}_{i^\vee}(s_\mu(\mathbf{x})) &= \begin{cases} s_{\mu \cup \{(i, \mu_i + 1)\}}(\mathbf{x}), & \text{if } \mu \cup \{(i, \mu_i + 1)\} \in \mathcal{P}, \\ 0, & \text{if } \mu \cup \{(i, \mu_i + 1)\} \notin \mathcal{P}, \end{cases} \\ \bar{u}_i(s_\mu(\mathbf{x})) &= \begin{cases} t s_{\mu \setminus \{(i, \mu_i)\}}(\mathbf{x}), & \text{if } \mu \setminus \{(i, \mu_i)\} \in \mathcal{P}, \\ 0, & \text{if } \mu \setminus \{(i, \mu_i)\} \notin \mathcal{P}. \end{cases}\end{aligned}$$

These operators are called Schur operators [3]. Let $\overline{\mathcal{U}}(t)$ be the subalgebra of $\text{End}_{\mathbb{Q}[t, t^{-1}]}(\Lambda(t))$ generated by $\bar{u}_i, \bar{u}_{i^\vee}$ ($i \geq 1$). It is easy to see that there exists a surjective $\mathbb{Q}[t, t^{-1}]$ -algebra homomorphism $\psi : \mathcal{U}(t) \rightarrow \overline{\mathcal{U}}(t)$ such that $\psi(u_i) = \bar{u}_i$ and $\psi(u_{i^\vee}) = \bar{u}_{i^\vee}$ for $i \in \mathbb{N}$.

For $\lambda \in \mathcal{P}$, let

$$s_\lambda(\bar{\mathbf{u}}) = \sum_{S \in \mathcal{B}_\lambda} \bar{u}_S, \quad s_\lambda^\vee(\bar{\mathbf{u}}) = \sum_{S \in \mathcal{B}_\lambda^\vee} \bar{u}_S,$$

where $\bar{u}_S = \psi(u_S)$ for $S \in \mathcal{B}_\lambda$ or \mathcal{B}_λ^\vee . For $\lambda, \mu \in \mathcal{P}$, we have

$$s_\mu^\vee(\bar{\mathbf{u}})(s_\lambda(\mathbf{x})) = s_\lambda(\mathbf{x})s_\mu(\mathbf{x}), \quad s_\mu(\bar{\mathbf{u}})(s_\lambda(\mathbf{x})) = t^{|\mu|} s_{\lambda/\mu}(\mathbf{x})$$

(see [3]). We also have

$$\begin{aligned}\overline{\mathcal{Q}}(\mathbf{x}_A) &= \prod_{a \in A}^{\rightarrow} \overline{\mathcal{Q}}(x_a) = \sum_{\lambda \in \mathcal{P}_A} s_\lambda(\bar{\mathbf{u}}) s_\lambda(\mathbf{x}_A), \\ \overline{\mathcal{P}}(\mathbf{x}_B) &= \prod_{b \in B}^{\rightarrow} \overline{\mathcal{P}}(x_b) = \sum_{\lambda \in \mathcal{P}_B} s_\lambda^\vee(\bar{\mathbf{u}}) s_\lambda(\mathbf{x}_B),\end{aligned}$$

where $\overline{\mathcal{P}}(x_a)$ and $\overline{\mathcal{Q}}(x_b)$ are obtained from $\mathcal{P}(x_a)$ and $\mathcal{Q}(x_b)$ by replacing u_i, u_{i^\vee} with $\bar{u}_i, \bar{u}_{i^\vee}$, respectively. Therefore, the products $\overline{\mathcal{Q}}(\mathbf{x}_A)\overline{\mathcal{P}}(\mathbf{x}_B)$ and $\overline{\mathcal{P}}(\mathbf{x}_B)\overline{\mathcal{Q}}(\mathbf{x}_A)$ are well defined, and the identity in Corollary 5.2 gives the following, which recovers the generalized Cauchy identity for Schur operators [3] when $t = 1$;

$$\overline{\mathcal{Q}}(\mathbf{x}_A)\overline{\mathcal{P}}(\mathbf{x}_B) = \overline{\mathcal{P}}(\mathbf{x}_B)\overline{\mathcal{Q}}(\mathbf{x}_A) \frac{\prod_{|a| \neq |b|} (1 + tx_a x_b)}{\prod_{|a| = |b|} (1 - tx_a x_b)}.$$

5.3. Knuth correspondence for skew tableaux. Fix $\alpha, \beta \in \mathcal{P}$. For $w = w_1 \dots w_r \in \mathcal{W}$, let us say that w is a Littlewood-Richardson (simply LR) word of shape (α, β) if there exist $\alpha^{(k)} \in \mathcal{P}$ for $1 \leq k \leq r$ such that $(w_1 \dots w_k \rightarrow H_\alpha) = H_{\alpha^{(k)}}$ and $\alpha^{(r)} = \beta$. Note that for $0 \leq k \leq r-1$, $|\alpha^{(k+1)}| = |\alpha^{(k)}| + 1$ if $w_k \in \mathbb{N}$ and $|\alpha^{(k+1)}| = |\alpha^{(k)}| - 1$ if $w_k \in \mathbb{N}^\vee$ (we assume that $\alpha^{(0)} = \alpha$), and by definition $w_1 \dots w_k$ is also a LR word of shape $(\alpha, \alpha^{(k)})$ for $1 \leq k \leq r$.

Lemma 5.3. *Let $\alpha, \beta \in \mathcal{P}$ be given. For $w \in \mathcal{W}$, w is a LR word of shape (α, β) if and only if $H_\alpha \otimes w \equiv H_\beta$. In particular, w' is a LR word of shape (α, β) for all $w' \in \mathcal{W}$ with $w' \equiv w$.*

Proof. Suppose that w is a LR word of shape (α, β) . Since $(w_1 \dots w_k \rightarrow H_\alpha) = H_\alpha \otimes w_1 \dots w_k$ for $1 \leq k \leq r$, it is clear that $H_\alpha \otimes w \equiv H_\beta$.

Conversely, suppose that $H_\alpha \otimes w \equiv H_\beta$. If $(w_1 \dots w_k \rightarrow H_\alpha) \in \mathcal{B}_{\mu, \nu}$ for some k and $\mu, \nu \in \mathcal{P}$ with $\nu \neq \emptyset$, then $(w \rightarrow H_\alpha) \in \mathcal{B}_{\sigma, \tau}$ for some $\sigma, \tau \in \mathcal{P}$ with $\tau \neq \emptyset$, which contradicts the fact that $H_\alpha \otimes w \equiv H_\beta$. Hence there exist $\alpha^{(k)} \in \mathcal{P}$ for $1 \leq k \leq r$ such that $(w_1 \dots w_k \rightarrow H_\alpha) \in \mathcal{B}_{\alpha^{(k)}}$ and $\alpha^{(r)} = \beta$.

Now suppose that $(w_1 \dots w_k \rightarrow H_\alpha) \neq H_{\alpha^{(k)}}$ for some $k \geq 1$, which is equivalent to saying that $\tilde{e}_i((w_1 \dots w_k \rightarrow H_\alpha)) \neq \mathbf{0}$ for some $i \geq 1$. Then $\tilde{e}_i H_\beta \equiv \tilde{e}_i(w \rightarrow H_\alpha) \equiv (\tilde{e}_i(w_1 \dots w_k \rightarrow H_\alpha)) \otimes w_{k+1} \dots w_r \neq \mathbf{0}$, which is also a contradiction. This completes the proof. \square

For $\lambda, \mu \in \mathcal{P}$ with $|\lambda| = |\alpha| + |\mu|$, we have by (2.1)

$$(5.1) \quad \{S \in \mathcal{B}_\mu \mid w_{\text{col}}(S) \text{ is a LR word of shape } (\alpha, \lambda)\} \xleftrightarrow{1-1} \mathbf{LR}_{\alpha\mu}^\lambda.$$

For $\lambda, \nu \in \mathcal{P}$ with $|\lambda| = |\beta| + |\nu|$, we have by (3.2)

$$(5.2) \quad \{S \in \mathcal{B}_\nu^\vee \mid w_{\text{col}}(S) \text{ is a LR word of shape } (\lambda, \beta)\} \xleftrightarrow{1-1} \mathbf{LR}_{\beta\nu}^\lambda.$$

Let $(\mathcal{B}_\mu \times \mathcal{B}_\nu^\vee)_{(\alpha, \beta)}$ be the set of $(S, T) \in \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee$ such that $w_{\text{col}}(S)w_{\text{col}}(T)$ is a LR word of shape (α, β) . Combining (5.1), (5.2) and Lemma 5.3, we have

$$(5.3) \quad (\mathcal{B}_\mu \times \mathcal{B}_\nu^\vee)_{(\alpha, \beta)} \xleftrightarrow{1-1} \bigsqcup_\lambda \mathbf{LR}_{\alpha\mu}^\lambda \times \mathbf{LR}_{\beta\nu}^\lambda.$$

Similarly, for $\sigma, \tau \in \mathcal{P}$, let $(\mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma)_{(\alpha, \beta)}$ be the set of $(S, T) \in \mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma$ such that $w_{\text{col}}(S)w_{\text{col}}(T)$ is a LR word of shape (α, β) . As in (5.3), we have a bijection

$$(5.4) \quad (\mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma)_{(\alpha, \beta)} \xleftrightarrow{1-1} \bigsqcup_\lambda \mathbf{LR}_{\lambda\tau}^\alpha \times \mathbf{LR}_{\lambda\sigma}^\beta.$$

Corollary 5.4. *Let $\alpha, \beta, \mu, \nu \in \mathcal{P}$ be given. The bijection (3.2) when restricted to $(\mathcal{B}_\mu \times \mathcal{B}_\nu^\vee)_{(\alpha, \beta)}$ gives the following bijection*

$$\bigsqcup_\lambda \mathbf{LR}_{\alpha\mu}^\lambda \times \mathbf{LR}_{\beta\nu}^\lambda \longrightarrow \bigsqcup_{\eta, \zeta, \sigma, \tau} \mathbf{LR}_{\eta\tau}^\alpha \times \mathbf{LR}_{\eta\sigma}^\beta \times \mathbf{LR}_{\sigma\zeta}^\mu \times \mathbf{LR}_{\tau\zeta}^\nu.$$

Proof. Since the bijection (3.2) preserves the plactic relations or the crystal equivalence, we have

$$(\mathcal{B}_\mu \times \mathcal{B}_\nu^\vee)_{(\alpha, \beta)} \longrightarrow \bigsqcup_{\zeta, \sigma, \tau} (\mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma)_{(\alpha, \beta)} \times \mathbf{LR}_{\sigma\zeta}^\mu \times \mathbf{LR}_{\tau\zeta}^\nu$$

Hence, it follows from (5.3) and (5.4). \square

Let $(\mathcal{M}_{\mathcal{A},\mathbb{N}} \times \mathcal{M}_{\mathcal{B},\mathbb{N}^\vee})_{(\alpha,\beta)}$ be the set of (A, A') such that $\mathbf{j} \cdot \mathbf{j}' \in \mathcal{W}$ is a LR word of shape (α, β) , where $A = A(\mathbf{i}, \mathbf{j})$ and $A' = A(\mathbf{i}', \mathbf{j}')$, and let $(\mathcal{M}_{\mathcal{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathcal{A},\mathbb{N}})_{(\alpha,\beta)}$ be defined in the same way.

Now, we recover the Knuth type correspondence for skew tableaux by Sagan and Stanley [16] as a restriction of Theorem 5.1 to LR words of shape (α, β) .

Theorem 5.5. *Let $\alpha, \beta \in \mathcal{P}$ be given. The bijection in Theorem 5.1 when restricted to $(\mathcal{M}_{\mathcal{A},\mathbb{N}} \times \mathcal{M}_{\mathcal{B},\mathbb{N}^\vee})_{(\alpha,\beta)}$ gives a bijection*

$$\bigsqcup_{\lambda} SST_{\mathcal{A}}(\lambda/\alpha) \times SST_{\mathcal{B}}(\lambda/\beta) \longrightarrow \bigsqcup_{\eta} SST_{\mathcal{A}}(\beta/\eta) \times SST_{\mathcal{B}}(\alpha/\eta) \times \mathcal{M}_{\mathcal{A},\mathcal{B}}.$$

Proof. Since the bijection in Theorem 5.1 preserves the plactic relations, we have a bijection

$$(5.5) \quad (\mathcal{M}_{\mathcal{A},\mathbb{N}} \times \mathcal{M}_{\mathcal{B},\mathbb{N}^\vee})_{(\alpha,\beta)} \longrightarrow (\mathcal{M}_{\mathcal{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathcal{A},\mathbb{N}})_{(\alpha,\beta)} \times \mathcal{M}_{\mathcal{A},\mathcal{B}}.$$

On the other hand, we have

$$\begin{aligned} & (\mathcal{M}_{\mathcal{A},\mathbb{N}} \times \mathcal{M}_{\mathcal{B},\mathbb{N}^\vee})_{(\alpha,\beta)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\mu,\nu} SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{B}}(\nu) \times (\mathcal{B}_{\mu} \times \mathcal{B}_{\nu}^{\vee})_{(\alpha,\beta)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda,\mu,\nu} SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{B}}(\nu) \times \mathbf{LR}_{\alpha\mu}^{\lambda} \times \mathbf{LR}_{\beta\nu}^{\lambda} \quad \text{by (5.3)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda,\mu,\nu} SST_{\mathcal{A}}(\mu) \times \mathbf{LR}_{\mu\alpha}^{\lambda} \times SST_{\mathcal{B}}(\nu) \times \mathbf{LR}_{\nu\beta}^{\lambda} \\ & \xleftrightarrow{1-1} \bigsqcup_{\lambda} SST_{\mathcal{A}}(\lambda/\alpha) \times SST_{\mathcal{B}}(\lambda/\beta) \quad \text{by (2.2).} \end{aligned}$$

Similarly, we have

$$\begin{aligned} & (\mathcal{M}_{\mathcal{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathcal{A},\mathbb{N}})_{(\alpha,\beta)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\sigma,\tau} SST_{\mathcal{A}}(\sigma) \times SST_{\mathcal{B}}(\tau) \times (\mathcal{B}_{\tau}^{\vee} \times \mathcal{B}_{\sigma})_{(\alpha,\beta)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\eta,\sigma,\tau} SST_{\mathcal{A}}(\sigma) \times SST_{\mathcal{B}}(\tau) \times \mathbf{LR}_{\eta\tau}^{\alpha} \times \mathbf{LR}_{\eta\sigma}^{\beta} \quad \text{by (5.4)} \\ & \xleftrightarrow{1-1} \bigsqcup_{\eta,\sigma,\tau} SST_{\mathcal{A}}(\sigma) \times \mathbf{LR}_{\sigma\eta}^{\beta} \times SST_{\mathcal{B}}(\tau) \times \mathbf{LR}_{\tau\eta}^{\alpha} \\ & \xleftrightarrow{1-1} \bigsqcup_{\eta} SST_{\mathcal{A}}(\beta/\eta) \times SST_{\mathcal{B}}(\alpha/\eta) \quad \text{by (2.2).} \end{aligned}$$

Combining with (5.5), we obtain the result. \square

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